ISOTROPIC IMMERSIONS

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1. Introduction

A Kaehler manifold of constant holomorphic sectional curvature is called a *complex space form*. A *Kaehler immersion* is an isometric immersion which is complex analytic. The second named author proved the following results.

Proposition 1 [2]. *Let M be an n-dimensional complex space form of constant holomorphic sectional curvature c, and $\tilde{M}$ be an $(n + p)$-dimensional complex space form of constant holomorphic sectional curvature $\tilde{c}$. If M is a Kaehler submanifold of $\tilde{M}$ with parallel second fundamental form, then either $c = \tilde{c}$ (i.e., M is totally geodesic in $\tilde{M}$) or $c = \frac{1}{2}\tilde{c}$, the latter case arising only when $\tilde{c} > 0$. Moreover, the immersion is rigid.*

Proposition 2 [3]. *Let M be an n-dimensional complex space form of constant holomorphic sectional curvature c, and $\tilde{M}$ be an $(n + \frac{1}{2}n(n + 1))$-dimensional complex space form of constant holomorphic sectional curvature $\tilde{c}$. If M is a Kaehler submanifold of $\tilde{M}$, then either $c = \tilde{c}$ (i.e., M is totally geodesic in $\tilde{M}$) or $c = \frac{1}{2}\tilde{c}$, the latter case arising only when $\tilde{c} > 0$. Moreover, the immersion is rigid.*

In the present paper, we shall prove similar results for real manifolds. An *isotropic immersion* is an isometric immersion such that all its normal curvature vectors have the same length at each point. A Riemannian manifold of constant curvature is called a *space form*.

Theorem 1. *Let M be an n-dimensional space form of constant curvature c, and $\tilde{M}$ be an $(n + \frac{1}{2}n(n + 1) - 1)$-dimensional space form of constant curvature $\tilde{c}$. If $c < \tilde{c}$, and M is an isotropic submanifold of $\tilde{M}$ with parallel second fundamental form, then $c = \dfrac{n}{2(n + 1)}\tilde{c}$, and the immersion is rigid.*

Theorem 2. *Let M be an n-dimensional space form of constant curvature c, and $\tilde{M}$ be an $(n + \frac{1}{2}n(n + 1) - 1)$-dimensional space form of constant curvature $\tilde{c}$. If $c < \tilde{c}$, and M is an isotropic submanifold of $\tilde{M}$, then $c = \dfrac{n}{2(n + 1)}\tilde{c}$, and the immersion is rigid provided that $n \leq 4$.*

Remark. Theorems 1 and 2 give a (local) characterization of a *Veronese manifold*.

2.   Preliminaries

Let M be an n-dimensional Riemannian manifold immersed isometrically in an $(n + p)$-dimensional space form $\tilde{M}$ of constant curvature $\tilde{c}$. We denote by ∇ (resp. $\tilde{\nabla}$) the covariant differentiation on M (resp. $\tilde{M}$). Then the second fundamental form σ of the immersion is given by

$$\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y ,$$

and satisfies $\sigma(X, Y) = \sigma(X, Y)$.

We choose a local field of orthonormal frames $e_1, \cdots, e_n, e_{\tilde{1}}, \cdots, e_{\tilde{p}}$ in $\tilde{M}$ in such a way that, restricted to M, $e_1, \cdots, e_n$ are tangent to M. With respect to the frame field of $\tilde{M}$ chosen above, let $\omega^1, \cdots, \omega^n, \omega^{\tilde{1}}, \cdots, \omega^{\tilde{p}}$ be the field of dual frames. Then the structure equations of $\tilde{M}$ are given by[1]

$$(2.1) \qquad d\omega^A = -\Sigma \omega_B^A \wedge \omega^B , \qquad \omega_B^A + \omega_A^B = 0 ,$$

$$(2.2) \qquad d\omega_B^A = -\Sigma \omega_C^A \wedge \omega_B^C + \tilde{c}\omega^A \wedge \omega^B .$$

Restricting these forms to M, we have the structure equations of the immersion:

$$(2.3) \qquad \omega^\alpha = 0 ,$$

$$(2.4) \qquad \omega_i^\alpha = \Sigma h_{ij}^\alpha \omega^j , \qquad h_{ij}^\alpha = h_{ji}^\alpha ,$$

$$(2.5) \qquad d\omega^i = -\Sigma \omega_j^i \wedge \omega^j , \qquad \omega_j^i + \omega_i^j = 0 ,$$

$$(2.6) \qquad d\omega_j^i = -\Sigma \omega_k^i \wedge \omega_j^k + \Omega_j^i , \qquad \Omega_j^i = \tfrac{1}{2}\Sigma R_{jkl}^i \omega^k \wedge \omega^l ,$$

$$(2.7) \qquad R_{jkl}^i = \tilde{c}(\delta_k^i \delta_{jl} - \delta_l^i \delta_{jk}) + \Sigma(h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha) .$$

The second fundamental form σ can be written as

$$(2.8) \qquad \sigma(e_i, e_j) = \Sigma h_{ij}^\alpha e_\alpha \quad \text{or} \quad \sigma = \Sigma h_{ij}^\alpha \omega^i \omega^j e_\alpha .$$

Define h_{ijk}^α by

$$(2.9) \qquad \Sigma h_{ijk}^\alpha \omega^k = dh_{ij}^\alpha - \Sigma h_{ik}^\alpha \omega_j^k - \Sigma h_{kj}^\alpha \omega_i^k + \Sigma h_{ij}^\beta \omega_\beta^\alpha .$$

Then from (2.2), (2.3) and (2.4) we have

$$(2.10) \qquad h_{ijk}^\alpha = h_{ikj}^\alpha .$$

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[1] We use the following convention on the ranges of indices unless otherwise stated:

$$A, B, C = 1, \cdots, n, \tilde{1}, \cdots, \tilde{p}; \qquad i, j, k, l = 1, \cdots, n;$$
$$a, b, c = 1, \cdots, n - 1; \qquad \alpha, \beta = \tilde{1}, \cdots, \tilde{p} .$$

The second fundamental form σ is said to be *parallel* if $h^\alpha_{ijk} = 0$ for all α, i, j and k. It is known that if the immersion is minimal, then the second fundamental form σ satisfies a differential equation. In fact, we have

Lemma 2.1 [1].

$$\tfrac{1}{2}\Delta(\Sigma h^\alpha_{ij} h^\alpha_{ij}) = \Sigma h^\alpha_{ijk} h^\alpha_{ijk} - 2\Sigma(h^\alpha_{ij} h^\alpha_{jk} h^\beta_{kl} h^\beta_{li} - h^\alpha_{ij} h^\beta_{jk} h^\alpha_{kl} h^\beta_{li})$$
$$- \Sigma h^\alpha_{ij} h^\beta_{ij} h^\alpha_{kl} h^\beta_{kl} + n\tilde{c}\Sigma h^\alpha_{ij} h^\alpha_{ij},$$

where Δ denotes the Laplacian.

3. Isotropically immersed space forms

For a unit vector X, $\sigma(X, X)$ is called the *normal curvature vector* determined by X. An isometric immersion is said to be *isotropic* if every normal curvature vector has the same length at each point. B. O'Neill [4] proved the following

Lemma 3.1. *Let M be an n-dimensional space form of constant curvature c, and $\tilde{M}$ be an $(n + \tfrac{1}{2}n(n + 1) - 1)$-dimensional space form of constant curvature $\tilde{c}$. If $c < \tilde{c}$, and M is an isotropic submanifold of $\tilde{M}$, then*

(i) M *is a minimal submanifold of $\tilde{M}$,*

(ii) $\|\sigma(X, X)\|^2 = \dfrac{2(n - 1)}{n + 2}(\tilde{c} - c)$ *for every unit vector X,*

(iii) $\|\sigma(X, Y)\|^2 = \dfrac{n}{n + 2}(\tilde{c} - c)$ *for every orthonormal pair X and Y,*

(iv)   *the $\tfrac{1}{2}n(n - 1)$ vectors $\sigma(e_i, e_j)$, $i < j$, are orthogonal,*

(v)   *the angle between $\sigma(e_i, e_i)$ and $\sigma(e_j, e_j)$ is the same (say θ) for every pair i and j ($i \neq j$) and $\cos\theta = -1/(n - 1)$,*

(vi) $\{\sigma(e_i, e_i)\}_{1 \leq i \leq n}$ *is orthogonal to $\{\sigma(e_i, e_j)\}_{1 \leq i < j \leq n}$,*

(vii)   *the dimension of the vector space generated by $\{\sigma(e_i, e_i)\}_{1 \leq i \leq n}$ and $\{\sigma(e_i, e_j)\}_{1 \leq i < j \leq n}$ is $\tfrac{1}{2}n(n + 1) - 1$.*

Let M be an n-dimensional space form of constant curvature c, and $\tilde{M}$ be an $(n + \tfrac{1}{2}n(n + 1) - 1)$-dimensional space form of constant curvature $\tilde{c}$. We assume that $c < \tilde{c}$, and that M is an isotropic submanifold of $\tilde{M}$.

Let $e_1, \cdots, e_n$ be a local field of orthonormal frames in M. From Lemma 3.1 we can see that the $\tfrac{1}{2}n(n + 1) - 1$ vectors $\sigma(e_a, e_a)$ and $\sigma(e_i, e_j)$, $1 \leq a \leq n - 1$, $1 \leq i < j \leq n$, form a basis of the normal space at each point of M. Using the Gram-Schmidt process, we can obtain an orthonormal basis of the normal space at each point of M. In fact, we have the following

Lemma 3.2. *The $\tfrac{1}{2}n(n + 1) - 1$ vectors*

$$\frac{\sqrt{n + 2}}{\sqrt{2n(n - a)(n - a + 1)(\tilde{c} - c)}} \left\{ \sum_{b=1}^{a-1} \sigma(e_b, e_b) + (n - a + 1)\sigma(e_a, e_a) \right\}$$

and $\dfrac{\sqrt{n + 2}}{\sqrt{n(\tilde{c} - c)}}\sigma(e_i, e_j)$ *for* $1 \leq a \leq n - 1$ *and* $1 \leq i < j \leq n$ *form an*

orthonormal system.

We choose a local field of orthonormal frames $e_1, \cdots, e_n, e_{\tilde{1}}, \cdots, e_{\tilde{p}}$ ($p = \frac{1}{2}n(n+1) - 1$) in $\tilde{M}$ in such a way that, restricted to M, $e_1, \cdots, e_n$ are tangent to M, and

$$e_{\tilde{a}} = \frac{\sqrt{n+2}}{\sqrt{2n(n-a)(n-a+1)(\tilde{c}-c)}} \left\{ \sum_{b=1}^{a-1} \sigma(e_b, e_b) + (n-a+1)\sigma(e_a, e_a) \right\},$$

$$e_{\widetilde{(i,j)}} = \frac{\sqrt{n+2}}{\sqrt{n(\tilde{c}-c)}}\sigma(e_i, e_j),$$

where $(i, j) = \min\{i, j\} + \frac{1}{2}|i - j|(2n + 1 - |i - j|) - 1$. With respect to this frame field, using **Lemma 3.1** we can obtain

$$(3.1) \quad (h_{ij}^a) = \begin{bmatrix} 0 & & & & 0 \\ & \ddots & & \vdots & \\ & & 0 & & \\ & \cdots & \lambda_a & \cdots & \\ & & \mu_a & & \\ & & \vdots & \ddots & \\ 0 & & & & \mu_a \end{bmatrix} a, \qquad (h_{ij}^{\widetilde{(i,j)}}) = \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & 0 & \cdots & \bigstar & \cdots \\ & \vdots & & \vdots & \\ \cdots & \bigstar & \cdots & 0 & \cdots \\ & \vdots & & \vdots & \end{bmatrix} \begin{matrix} i \\ \\ j \end{matrix},$$

where

$$\lambda_a = \frac{\sqrt{2n(n-a)(\tilde{c}-c)}}{\sqrt{(n+2)(n-a+1)}}, \qquad \mu_a = -\frac{\sqrt{2n(\tilde{c}-c)}}{\sqrt{(n+2)(n-a)(n-a+1)}},$$

$$\bigstar = \sqrt{n(\tilde{c}-c)}/\sqrt{n+2}$$

Thus from (3.1), Lemma 2.1 and Lemma 3.1 we have

Lemma 3.3. *Let M be an n-dimensional space form of constant curvature c, and $\tilde{M}$ be an $(n + \frac{1}{2}n(n+1) - 1)$-dimensional space form of constant curvature $\tilde{c}$. If $c < \tilde{c}$, and M is an isotropic submanifold of $\tilde{M}$, then*

$$\Sigma h_{ijk}^\alpha h_{ijk}^\alpha = \frac{2n^2(n^2-1)}{n+2}(\tilde{c}-c)\left\{\frac{n}{2(n+1)}\tilde{c} - c\right\}.$$

We need as well the following

Lemma 3.4. *Let M be an n-dimensional space form of constant curvature c, and $\tilde{M}$ be an $(n + \frac{1}{2}n(n+1) - 1)$-dimensional space form of constant curvature $\tilde{c}$. Suppose $c < \tilde{c}$, and M is an isotropic submanifold of $\tilde{M}$. Then the second fundamental form σ is parallel if and only if the following hold:*

$$(3.2) \qquad \omega_{\tilde{b}}^{\tilde{a}} = 0 \ , \qquad \omega_{(\widetilde{a,b})}^{a} = \frac{\sqrt{2(n-a)}}{\sqrt{n-a+1}} \omega_b^a \quad (b < a) \ ,$$

$$(3.3) \qquad \omega_{(\widetilde{a,j})}^{\tilde{a}} = \frac{\sqrt{2(n-a+1)}}{\sqrt{n-a}} \omega_j^a \qquad (a < j) \ ,$$

$$(3.4) \qquad \omega_{(\widetilde{j,k})}^{\tilde{a}} = 0 \qquad (j, k > a \quad or \quad < a) \ ,$$

$$(3.5) \qquad \omega_{(\widetilde{j,k})}^{\tilde{a}} = \frac{\sqrt{2}}{\sqrt{(n-a)(u-a+1)}} \omega_k^j \qquad (j < a < k) \ ,$$

$$(3.6) \qquad \omega_{(\widetilde{i,k})}^{(i,j)} = \omega_k^j \ ,$$

$$(3.7) \qquad \omega_{(\widetilde{k,l})}^{(\widetilde{i,j})} = 0 \ ,$$

where different indices indicate different numbers.

Proof. From (2.9) and (3.1) it follows that the second fundamental form σ is parallel if and only if (3.2), $\cdots$, (3.7) and the following equations hold:

$$(3.8) \qquad \frac{\sqrt{n-b}}{\sqrt{n-b+1}} \omega_b^{\tilde{a}} = \sum_{c < b} \frac{1}{\sqrt{(n-c)(n-c+1)}} \omega_{\tilde{c}}^{\tilde{a}} \ ,$$

$$(3.9) \qquad \sum_{c < a} \frac{1}{\sqrt{(n-c)(n-c+1)}} \omega_{\tilde{c}}^{\tilde{a}} = 0 \ ,$$

$$(3.10) \qquad \frac{\sqrt{n-a}}{\sqrt{n-a+1}} \omega_{(\widetilde{a,j})}^{\tilde{a}} + \sum_{k < a} \frac{1}{\sqrt{(n-k)(n-k+1)}} \omega_{\tilde{k}}^{(\widetilde{a,j})} = \sqrt{2} \, \omega_j^a \ ,$$

$$(3.11) \qquad \frac{\sqrt{n-b}}{\sqrt{n-b+1}} \omega_{\tilde{b}}^{(i,j)} = \sum_{c < b} \frac{1}{\sqrt{(n-c)(n-c+1)}} \omega_{\tilde{c}}^{(i,j)} \ ,$$

where different indices indicate different numbers. We can see inductively that (3.8) and (3.9) are equivalent to $\omega_{\tilde{b}}^{\tilde{a}} = 0$. Moreover, (3.2), $\cdots$, (3.5) imply (3.10) and (3.11).   q.e.d.

From (2.2), (2.3), (2.4) and (3.1) we have

$$(3.12) \qquad \begin{aligned} &\left(\sum_{b < a} \frac{\sqrt{2}}{\sqrt{(n-b)(n-b+1)}} \omega_b^{\tilde{a}} \right) \wedge \omega^a \\ &+ \sum_{b < a} \left(\frac{\sqrt{2(n-a)}}{\sqrt{n-a+1}} \omega_b^a - \omega_{(\widetilde{a,b})}^{\tilde{a}} \right) \wedge \omega^b \\ &+ \sum_{a < b} \left(\frac{\sqrt{2(n-a+1)}}{\sqrt{n-a}} \omega_b^a - \omega_{(\widetilde{a,b})}^{\tilde{a}} \right) \wedge \omega^b = 0 \ , \end{aligned}$$

$$\left(\frac{\sqrt{2(n-a)}}{\sqrt{n-a+1}}\omega_b^a - \omega_{(a,b)}^{\tilde{a}}\right) \wedge \omega^a$$

$$(3.13) \qquad -\left(\frac{\sqrt{2(n-b)}}{\sqrt{n-b+1}}\omega_b^{\tilde{a}} - \sum_{c<b}\frac{\sqrt{2}}{\sqrt{(n-c)(n-c+1)}}\omega_{\tilde{c}}^{\tilde{a}}\right) \wedge \omega^b$$

$$-\sum_{c<a}\omega_{(b,c)}^{\tilde{a}} \wedge \omega^c + \sum_{a<k}\left(\frac{\sqrt{2}}{\sqrt{(n-a)(n-a+1)}}\omega_k^b - \omega_{(b,k)}^{\tilde{a}}\right) \wedge \omega^k$$

$$= 0 \qquad (b<a) ,$$

$$\left(\frac{\sqrt{2(n-a+1)}}{\sqrt{n-a}}\omega_a^j - \omega_{(a,j)}^{\tilde{a}}\right) \wedge \omega^a$$

$$(3.14) \qquad +\left(\frac{\sqrt{2(n-j)}}{\sqrt{n-j+1}}\omega_j^{\tilde{a}} - \sum_{k<j}\frac{\sqrt{2}}{\sqrt{(n-k)(n-k+1)}}\omega_{\tilde{k}}^{\tilde{a}}\right) \wedge \omega^j$$

$$+\sum_{c<a}\left(\frac{\sqrt{2}}{\sqrt{(n-a)(n-a+1)}}\omega_c^j + \omega_{(j,c)}^{\tilde{a}}\right) \wedge \omega^c + \sum_{a<k}\omega_{(j,c)}^{\tilde{a}} \wedge \omega^c$$

$$= 0 \qquad (a<j) ,$$

$$(3.15) \quad \left(\frac{\sqrt{2(n-i)}}{\sqrt{n-i+1}}\omega_i^{(i,j)} - \sum_{k<i}\frac{\sqrt{2}}{\sqrt{(n-k)(n-k+1)}}\omega_k^{(i,j)} - 2\omega_i^j\right) \wedge \omega^i$$

$$+ \sum (\omega_{(i,k)}^{(i,j)} - \omega_k^j) \wedge \omega^k = 0 ,$$

$$(\omega_{(i,k)}^{(i,j)} - \omega_k^j) \wedge \omega^i + (\omega_{(k,j)}^{(i,j)} - \omega_k^i) \wedge \omega^j$$

$$(3.16) \qquad + \left(\frac{\sqrt{2(n-k)}}{\sqrt{n-k+1}}\omega_k^{(i,j)} - \sum_{l<k}\frac{\sqrt{2}}{\sqrt{(n-l)(n-l+1)}}\omega_l^{(i,j)}\right) \wedge \omega^k$$

$$+ \sum_{l\neq i,j,k}\omega_{(k,l)}^{(i,j)} \wedge \omega^l = 0 .$$

By (3.15) and Cartan's lemma we may write $\omega_{(i,k)}^{(i,j)} - \omega_k^j = \Sigma A_{kl}^{ij}\omega^l$, where $A_{kl}^{ij} = A_{lk}^{ij}$. Since $\omega_{(i,k)}^{(i,j)} - \omega_k^j + \omega_{(i,j)}^{(i,k)} - \omega_j^k = 0$ so that $A_{kl}^{ij} + A_{jl}^{ik} = 0$, we can see $A_{kl}^{ij} = 0$. Hence we have

$$(3.17) \qquad\qquad\qquad \omega_{(i,k)}^{(i,j)} = \omega_k^j .$$

From (3.16) and (3.17) it follows that $\omega_{(k,l)}^{(i,j)}$ contain neither ω^i and ω^j nor ω^k and ω^l by symmetry. Therefore

$$(3.18) \qquad\qquad \omega_{(k,l)}^{(i,j)} \ \text{do not contain} \ \omega^i, \ \omega^j, \ \omega^k \ \text{and} \ \omega^l .$$

4. Proofs of theorems

Theorem 1 follows immediately from Lemmas 3.3 and 3.4. We shall give here a proof of Theorem 2 for $n = 4$. The proof of Theorem 2 for $n = 2$ and $n = 3$ is quite similar to and easier than that for $n = 4$.

In consideration of (3.17) and (3.18), equations $(3.12), \cdots, (3.16)$ can be written as follows:

$$\left(\frac{2\sqrt{2}}{\sqrt{3}}\omega_2^1 - \omega_4^{\bar{1}}\right) < \omega^2 + \left(\frac{2\sqrt{2}}{\sqrt{3}}\omega_3^1 - \omega_7^{\bar{1}}\right) \wedge \omega^3$$

$$+ \left(\frac{2\sqrt{2}}{\sqrt{3}}\omega_4^1 - \omega_9^{\bar{1}}\right) \wedge \omega^4 = 0 ,$$

(4.1)
$$\left(\frac{2}{\sqrt{3}}\omega_1^2 - \omega_4^{\bar{2}}\right) \wedge \omega^1 + \frac{1}{\sqrt{6}}\omega_1^{\bar{2}} \wedge \omega^2 + (\sqrt{3}\,\omega_3^2 - \omega_{\bar{5}}^{\bar{2}}) \wedge \omega^3$$

$$+ (\sqrt{3}\,\omega_4^2 - \omega_8^{\bar{2}}) \wedge \omega^4 = 0 ,$$

$$(\omega_1^3 - \omega_7^{\bar{3}}) \wedge \omega^1 + (\omega_2^3 - \omega_5^{\bar{3}}) \wedge \omega^2 + \left(\frac{1}{\sqrt{6}}\omega_1^3 + \frac{1}{\sqrt{3}}\omega_2^{\bar{3}}\right) \wedge \omega^3$$

$$+ (2\omega_4^3 - \omega_6^{\bar{3}}) \wedge \omega^4 = 0 ;$$

$$-\frac{\sqrt{3}}{\sqrt{2}}\omega_1^{\bar{2}} \wedge \omega^1 + \left(\frac{\sqrt{2}}{\sqrt{3}}\omega_1^2 - \omega_4^{\bar{2}}\right) \wedge \omega^2 + \left(\frac{1}{\sqrt{3}}\omega_3^2 - \omega_7^{\bar{2}}\right) \wedge \omega^3$$

$$+ \left(\frac{1}{\sqrt{3}}\omega_4^2 - \omega_8^{\bar{2}}\right) \wedge \omega^4 = 0 ,$$

(4.2)
$$-\frac{\sqrt{3}}{\sqrt{2}}\omega_1^3 \wedge \omega^1 - \omega_4^{\bar{3}} \wedge \omega^2 + (\omega_1^3 - \omega_7^{\bar{3}}) \wedge \omega^3$$

$$+ (\omega_4^1 - \omega_6^{\bar{3}}) \wedge \omega^4 = 0 ,$$

$$-\omega_4^{\bar{3}} \wedge \omega^1 - \left(\frac{\sqrt{2}}{\sqrt{3}}\omega_2^3 - \frac{1}{\sqrt{6}}\omega_1^3\right) \wedge \omega^2 + (\omega_2^3 - \omega_5^{\bar{3}}) \wedge \omega^3$$

$$+ (\omega_4^2 - \omega_8^{\bar{3}}) \wedge \omega^4 = 0 ;$$

$$\left(\frac{2\sqrt{2}}{\sqrt{3}}\omega_1^2 + \omega_4^{\bar{1}}\right) \wedge \omega^1 + \frac{2}{\sqrt{3}}\omega_2^{\bar{1}} \wedge \omega^2 + \omega_5^{\bar{1}} \wedge \omega^3 + \omega_8^{\bar{1}} \wedge \omega^4 = 0 ,$$

$$\left(\frac{2\sqrt{2}}{\sqrt{3}}\omega_1^3 + \omega_7^{\bar{1}}\right) \wedge \omega^1 + \omega_5^{\bar{1}} \wedge \omega^2 + \left(\omega_3^{\bar{1}} - \frac{1}{\sqrt{3}}\omega_2^{\bar{1}}\right) \wedge \omega^3$$

$$+ \omega_8^{\bar{1}} \wedge \omega^4 = 0 ,$$

$$\left(\frac{2\sqrt{2}}{\sqrt{3}}\omega_1^4 + \omega_9^{\bar{1}}\right) \wedge \omega^1 + \omega_8^{\bar{1}} \wedge \omega^2 + \omega_6^{\bar{1}} \wedge \omega^3$$

$$- \left(\frac{1}{\sqrt{3}}\omega_2^{\bar{1}} + \omega_3^{\bar{1}}\right) \wedge \omega^4 = 0 ,$$

$$\left(\frac{1}{\sqrt{3}}\omega_1^3 + \omega_7^{\bar{3}}\right) \wedge \omega^1 + (\sqrt{3}\,\omega_2^3 + \omega_5^{\bar{3}}) \wedge \omega^2 + \left(\omega_3^{\bar{3}} - \frac{1}{\sqrt{6}}\omega_1^{\bar{3}}\right) \wedge \omega^3$$

(4.3)
$$+ \; \omega_6^{\bar{3}} \wedge \omega^4 = 0 \, ,$$

$$\left(\frac{1}{\sqrt{3}}\omega_1^4 + \omega_8^{\bar{3}}\right) \wedge \omega^1 + (\sqrt{3}\,\omega_2^4 + \omega_6^{\bar{3}}) \wedge \omega^2 + \omega_8^{\bar{3}} \wedge \omega^3$$

$$- \left(\frac{1}{\sqrt{6}}\omega_1^{\bar{3}} + \omega_3^{\bar{3}}\right) \wedge \omega^4 = 0 \, ,$$

$$(\omega_1^4 + \omega_8^{\bar{3}}) \wedge \omega^1 + (\omega_2^4 + \omega_6^{\bar{3}}) \wedge \omega^2 + (2\omega_3^4 + \omega_6^{\bar{3}}) \wedge \omega^3$$

$$\div \left(\frac{1}{\sqrt{6}}\omega_1^{\bar{3}} + \frac{1}{\sqrt{3}}\omega_2^{\bar{3}}\right) \wedge \omega^4 = 0;$$

$$\left(\frac{\sqrt{3}}{\sqrt{2}}\omega_1^{\bar{4}} - 2\omega_1^2\right) \wedge \omega^1 = 0 \, ,$$

$$\left(\frac{\sqrt{3}}{\sqrt{2}}\omega_1^{\bar{7}} - 2\omega_1^3\right) \wedge \omega^1 = 0 \, ,$$

$$\left(\frac{\sqrt{3}}{\sqrt{2}}\omega_1^{\bar{5}} - 2\omega_1^4\right) \wedge \omega^1 = 0 \, ,$$

$$\left(\frac{2}{\sqrt{3}}\omega_2^{\bar{4}} - \frac{1}{\sqrt{6}}\omega_1^{\bar{4}} - 2\omega_2^1\right) \wedge \omega^2 = 0 \, ,$$

$$\left(\frac{2}{\sqrt{3}}\omega_2^{\bar{5}} - \frac{1}{\sqrt{6}}\omega_1^{\bar{5}} - 2\omega_2^3\right) \wedge \omega^2 = 0 \, ,$$

$$\left(\frac{2}{\sqrt{3}}\omega_2^{\bar{6}} - \frac{1}{\sqrt{6}}\omega_1^{\bar{6}} - 2\omega_2^4\right) \wedge \omega^2 = 0 \, ,$$

(4.4)
$$\left(\omega_3^{\bar{7}} - \frac{1}{\sqrt{6}}\omega_1^{\bar{7}} - \frac{1}{\sqrt{3}}\omega_2^{\bar{7}} - 2\omega_3^1\right) \wedge \omega^3 = 0 \, ,$$

$$\left(\omega_3^{\bar{5}} - \frac{1}{\sqrt{6}}\omega_1^{\bar{5}} - \frac{1}{\sqrt{3}}\omega_2^{\bar{5}} - 2\omega_3^2\right) \wedge \omega^3 = 0 \, ,$$

$$\left(\omega_3^{\bar{6}} - \frac{1}{\sqrt{6}}\omega_1^{\bar{6}} - \frac{1}{\sqrt{3}}\omega_2^{\bar{6}} - 2\omega_3^4\right) \wedge \omega^3 = 0 \, ,$$

$$\left(\frac{1}{\sqrt{6}}\omega_1^{\bar{5}} + \frac{1}{\sqrt{3}}\omega_2^{\bar{5}} + \omega_3^{\bar{5}} + 2\omega_4^1\right) \wedge \omega^4 = 0 \, ,$$

$$\left(\frac{1}{\sqrt{6}}\omega_1^{\bar{5}} + \frac{1}{\sqrt{3}}\omega_2^{\bar{5}} + \omega_3^{\bar{5}} + 2\omega_4^2\right) \wedge \omega^4 = 0 \, ,$$

$$\left(\frac{1}{\sqrt{6}}\omega_1^{\bar{6}} + \frac{1}{\sqrt{3}}\omega_2^{\bar{6}} + \omega_3^{\bar{6}} + 2\omega_4^3\right) \wedge \omega^4 = 0;$$

$$\left(\omega_3^{\bar{4}} - \frac{1}{\sqrt{6}}\omega_1^{\bar{4}} - \frac{1}{\sqrt{3}}\omega_2^{\bar{4}}\right) \wedge \omega^3 = 0 \ ,$$

$$\left(\frac{1}{\sqrt{6}}\omega_1^{\bar{4}} + \frac{1}{\sqrt{3}}\omega_2^{\bar{4}} + \omega_3^{\bar{4}}\right) \wedge \omega^4 = 0 \ ,$$

$$\left(\frac{2}{\sqrt{3}}\omega_2^{\bar{7}} - \frac{1}{\sqrt{6}}\omega_1^{\bar{7}}\right) \wedge \omega^2 = 0 \ ,$$

$$\left(\frac{1}{\sqrt{6}}\omega_1^{\bar{7}} + \frac{1}{\sqrt{3}}\omega_2^{\bar{7}} + \omega_3^{\bar{7}}\right) \wedge \omega^4 = 0 \ ,$$

(4.5) $$\left(\frac{2}{\sqrt{3}}\omega_2^{\bar{5}} - \frac{1}{\sqrt{6}}\omega_1^{\bar{5}}\right) \wedge \omega^2 = 0 \ ,$$

$$\left(\omega_3^{\bar{5}} - \frac{1}{\sqrt{6}}\omega_1^{\bar{5}} - \frac{1}{\sqrt{3}}\omega_2^{\bar{5}}\right) \wedge \omega^3 = 0 \ , \quad \omega_1^{\bar{2}} \wedge \omega^1 = 0 \ ,$$

$$\left(\frac{1}{\sqrt{6}}\omega_1^{\bar{5}} + \frac{1}{\sqrt{3}}\omega_2^{\bar{5}} + \omega_3^{\bar{5}}\right) \wedge \omega^4 = 0 \ , \quad \omega_1^{\bar{5}} \wedge \omega^1 = 0 \ ,$$

$$\left(\omega_3^{\bar{8}} - \frac{1}{\sqrt{6}}\omega_1^{\bar{8}} - \frac{1}{\sqrt{3}}\omega_2^{\bar{8}}\right) \wedge \omega^3 = 0 \ , \quad\quad \omega_1^{\bar{6}} \wedge \omega^1 = 0 \ ,$$

$$\left(\frac{2}{\sqrt{3}}\omega_2^{\bar{6}} - \frac{1}{\sqrt{6}}\omega_1^{\bar{6}}\right) \wedge \omega^2 = 0 \ .$$

From $(4.1)_1$, $(4.4)_1, \cdots, (4.4)_3$; $(4.3)_2$, $(4.5)_7$, (4.7); $(4.3)_3$, $(4.5)_{11}$, (4.8); $(4.3)_1$, $(4.5)_9$, (4.6); $(4.3)_1$, (4.6), (4.9), (4.10); $(4.3)_2$, (4.7), (4.9), (4.10); $(4.3)_3$, (4.8), (4.10), (4.11); $(4.12), \cdots, (4.14)$; $(4.15)_{12}$, (4.10); $(4.4)_5$, (4.9); $(4.4)_6$, (4.11); $(4.3)_5$, (4.16), (4.18); $(4.4)_9$, $(4.4)_{12}$, (4.10), (4.19); $(4.1)_2$, (4.12), (4.17); $(4.1)_2$, (4.12), (4.18), (4.21); we obtain, respectively,

(4.6) $$\omega_4^{\bar{1}} = \frac{2\sqrt{2}}{\sqrt{3}}\omega_2^1 \ ,$$

(4.7) $$\omega_7^{\bar{1}} = \frac{2\sqrt{2}}{\sqrt{3}}\omega_3^1 \ ,$$

(4.8) $$\omega_8^{\bar{1}} = \frac{2\sqrt{2}}{\sqrt{3}}\omega_4^1 \ ;$$

(4.9) $$\omega_5^{\bar{1}} = 0 \ ;$$

(4.10) $$\omega_6^{\bar{1}} = 0 \ ;$$

(4.11) $$\omega_8^{\bar{1}} = 0 \ ;$$

(4.12) $$\omega_2^{\bar{1}} \in \{\{\omega^2\}\} \ ,$$

" $\in \{\{\cdots\}\}$ " indicating "is a linear combination of $\cdots$ ";

$$(4.13) \qquad \sqrt{3}\,\omega_3^{\bar{1}} - \omega_2^{\bar{1}} \in \{\{\omega^3\}\}\,;$$

$$(4.14) \qquad \omega_2^{\bar{1}} + \sqrt{3}\,\omega_3^{\bar{1}} \in \{\{\omega^4\}\}\,;$$

$$(4.15) \qquad \omega_2^{\bar{1}} = 0\,, \qquad \omega_3^{\bar{1}} = 0\,;$$

$$(4.16) \qquad \omega_8^{\bar{2}} \in \{\{\omega^2\}\}\,;$$

$$(4.17) \qquad \omega_5^{\bar{2}} - \sqrt{3}\,\omega_3^2 \in \{\{\omega^2\}\}\,;$$

$$(4.18) \qquad \omega_8^{\bar{2}} - \sqrt{3}\,\omega_4^2 \in \{\{\omega^2\}\}\,;$$

$$(4.19) \qquad \omega_8^{\bar{2}} = 0\,;$$

$$(4.20) \qquad \omega_8^{\bar{3}} - 2\omega_4^3 = 0\,;$$

$$(4.21) \qquad \omega_5^{\bar{3}} - \sqrt{3}\,\omega_3^2 = 0\,;$$

$$(4.22) \qquad \omega_8^{\bar{3}} - \sqrt{3}\,\omega_4^2 = 0\,;$$

and hence

$$(4.23) \qquad \sqrt{3}\,\omega_4^{\bar{3}} - 2\omega_1^2 \in \{\{\omega^1\}\}\,.$$

From $(4.2)_1$, (4.15), (4.23); $(4.3)_4$, $(4.3)_5$, (4.15), (4.17), $\cdots$, (4.19); $(4.2)_1$, $(4.3)_4$, (4.15), $\cdots$, (4.17), (4.23); $(4.2)_1$, $(4.3)_5$, (4.15), (4.16), (4.18), (4.23); $(4.4)_8$, (4.9), (4.21); $(4.4)_{11}$, (4.11), (4.22); $(4.2)_2$, $(4.2)_3$, (4.15), (4.28), (4.29); we have, respectively,

$$(4.24) \qquad \sqrt{3}\,\omega_4^{\bar{3}} = 2\omega_1^2\,;$$

$$(4.25) \qquad \omega_3^{\bar{2}} \in \{\{\omega^1\}\}\,;$$

$$(4.26) \qquad \sqrt{3}\,\omega_7^{\bar{2}} - \omega_3^1 \in \{\{\omega^3\}\}\,;$$

$$(4.27) \qquad \sqrt{3}\,\omega_8^{\bar{2}} - \omega_4^1 \in \{\{\omega^4\}\}\,;$$

$$(4.28) \qquad \omega_5^{\bar{3}} - \omega_2^3 \in \{\{\omega^3\}\}\,;$$

$$(4.29) \qquad \omega_8^{\bar{3}} - \omega_4^2 \in \{\{\omega^4\}\}\,;$$

$$(4.30) \qquad \omega_4^{\bar{3}} \in \{\{\omega^2\}\}\,.$$

From $(4.5)_1$ and $(4.5)_2$ it follows that $\omega_4^{\bar{3}} \in \{\{\omega^3, \omega^4\}\}$ which, together with (4.30), implies

(4.31) $$\omega_4^{\bar{3}} = 0 \ .$$

From $(4.2)_3$, (4.15), (4.28), (4.29) and (4.31) we obtain $\omega_3^{\bar{2}} \in \{\{\omega^2\}\}$ which, together with (4.25), implies

(4.32) $$\omega_3^{\bar{2}} = 0 \ .$$

From $(4.1)_3$, $(4.2)_2$, (4.15), (4.20), (4.30) and (4.32) we have

(4.33) $$\omega_7^{\bar{3}} - \omega_1^3 = 0 \ ,$$

and hence

(4.34) $$\omega_5^{\bar{3}} - \omega_2^3 \in \{\{\omega^2\}\} \ ,$$

(4.35) $$\omega_9^{\bar{3}} - \omega_4^1 \in \{\{\omega^4\}\} \ .$$

From (4.28), (4.34); $(4.3)_4$, (4.15), (4.16), (4.21), (4.32); we obtain, respectively,

(4.36) $$\omega_5^{\bar{3}} - \omega_2^3 = 0;$$

and $\sqrt{3}\,\omega_7^{\bar{2}} - \omega_3^1 \in \{\{\omega^1\}\}$ which, together with (4.26), implies

(4.37) $$\sqrt{3}\,\omega_7^{\bar{2}} - \omega_3^1 = 0 \ .$$

From $(4.3)_5$, (4.15), (4.16), (4.18) and (4.32) we have $\sqrt{3}\,\omega_9^{\bar{2}} - \omega_4^1 \in \{\{\omega^1\}\}$ which, together with (4.27), implies

(4.38) $$\sqrt{3}\,\omega_9^{\bar{2}} - \omega_4^1 = 0 \ .$$

From $(4.3)_6$, (4.15), (4.20), (4.29), (4.32) and (4.35) we obtain

(4.39) $$\omega_9^{\bar{3}} - \omega_4^1 = 0 \ ,$$

(4.40) $$\omega_8^{\bar{3}} - \omega_4^2 = 0 \ .$$

Now it is easy to see that (3.17), (3.18), (4.6), $\cdots$, (4.11), (4.15), (4.19), $\cdots$, (4.22), (4.24), (4.31), $\cdots$, (4.33) and (4.36), $\cdots$, (4.40), together with Lemma 3.4, imply that the second fundamental form is parallel. This, combined with Theorem 1, thus gives Theorem 2.

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